

Euler Top dynamics of Nambu-Goto p-branes

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ABSTRACT: We propose a method to obtain new exact solutions of spinning p-branes in flat space-times for any p, which manifest themselves as higher dimensional Euler Tops and minimize their energy functional. We provide concrete examples for the case of spherical topology S^2, S^3 and rotational symmetry $\prod_i SO(q_i)$. In the case of toroidal topology T^2, T^3 the rotational symmetry is $\prod_i SU(q_i)$ with m target dimensions being compactified on the torus T^m . By double dimensional reduction the Light Cone Hamiltonians of T^2, T^3 reduce to those of closed string S^1 and T^2 membranes respectively. The solutions are interpreted as non-perturbative spinning soliton states of type *IIA – IIB* superstrings.

KEYWORDS: p-branes, Brane Dynamics in Gauge Theories, M(atrix) Theories, Gauge-gravity correspondence.

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1. Introduction

One of the most important discoveries in theoretical physics in the last few years has been the connection of the strongly coupled gauge theories to perturbative gravity through the Maldacena conjecture [1]. This is only one spectacular result of the UV/IR relation and Holography, discovered in non-commutative geometry of D-branes in gravitational backgrounds with fluxes [2]. In order to understand this connection, the most important tool has been the comparison of the energy spectra of rotating strings, D-branes, p-branes and/or even matrix model rotating solutions in various gravitational backgrounds with the anomalous dimensions of composite operators of the boundary gauge, or more generally, of the conformal field theory. More recently such a comparison has been in the focus due to their connection with Bethe ansatz methods of obtaining the spectra of integrable spin chain models. Impressive agreement on both sides has been obtained [3].

Another interesting development has come about by the use of rotating D_3 branes in the presence of fluxes giving rise to a stringy exclusion principle as well as the notion of giant graviton [1, 4]. Rotating solutions in backgrounds of pp waves along with their dielectric behaviour in the presence of fluxes has been studied. Their connection with the BPS sector of $N = 4$ Super-Yang-Mills theories has been established [5]. In a completely different direction Matrix or brane solutions have been interpreted in the framework of Matrix Cosmology [6]. An important class of new nonrelativistic Newton-Hook cosmologies appears from deSitter spacetime backgrounds in the Newton-Hooke limit of $\frac{\Lambda c^2}{3} = \text{constant}$ as $\Lambda \rightarrow 0$ and $c^2 \rightarrow \infty$ [7].

Rotating Solutions for strings and p-branes were studied in the first few years of the development of this field by searching for massless particles in their spectra [8]. In the case of superstring theory the full supergravity multiplets have been discovered raising, as a consequence, the string to the status of a more fundamental theory. Much later it was understood that other extended objects, such as D-branes [9] are connected through nonperturbative dualities. This has led to the creation of the hypothesis of M-theory and Matrix model [10].

It is obvious from the above that there is a strong motivation for a more exhaustive search for non-perturbative soliton solutions of string theories such as membranes, 3-branes and/or matrix model solutions in various backgrounds with or without fluxes. All of these should be compared with known spectra of operators of gauge or conformal field theories. Another interesting application can be the determination of the quantum effective Hamiltonian for p-branes as fundamental objects [11].

In this paper we propose a method to extract new solutions for spinning p-branes in the Light Cone spacetime for any p. By providing concrete examples we continue our search for membrane or matrix solutions [12] in a more systematic way, thus exhausting the class of rotating solutions for S^2, T^2 in flat space times with toroidal compactifications which are consistent with our method. We demonstrate that the rigid body type of Eulerian motion minimizes the energy with a given conserved angular momentum. We extend these solutions to higher dimensionalities of the extended object (e.g. $p = 3$ for S^3, T^3). The method can be applied to any p. In order to achieve this we make use of the lightcone gauge where Nambu brackets play a natural role by expressing the extension of the infinite gauge group from area preserving diffeomorphisms ($p = 2$) to p-volume preserving diffeomorphisms.

Although for $p \geq 3$ there have been efforts to formulate corresponding matrix models [13] we will not attempt to apply our method to these models. We believe that if fluxes are not present (absence of Dielectric Myers effect) matrix models fuzzify only membranes ($p = 2$) because of the generic two discrete indices of matrices. Higher values of p which constitute generalizations to multiindexed matrices with p discrete indices are necessary. Unknown mathematical structures for multiplication and more general algebraic operations of these objects must be sought for.

We will restrict ourselves to flat backgrounds with toroidal compactifications. We observe that the world volumes of our solutions live in submanifolds with spherical or toroidal geometry. This property may possibly be used to embed isometrically our solutions into curved spacetime backgrounds with the same world volumes as minimal submanifolds. These embeddings might provide solutions of the extended objects in these specific curved backgrounds (e.g. $AdS^5 \times S^5, AdS^7 \times S^4$ and G^2) [14].

We organize our work as follows:

In section 2 we write down the equations of motion and their constraints in the lightcone gauge for p-branes and the matrix model in flat spacetimes. We introduce the Nambu brackets, a minimum set of their properties as well as the definitions of their p-volume preserving diffeomorphisms.

In section 3 we construct the extension of the Euler Top equations of motion to higher dimensions which are appropriate for p-branes. We write the relation between their total

energy, angular momenta and generalized angular velocities. We provide the NASCs in order that a p-brane solution can be characterized as higher dim. Euler Top ("P-Branetops").

In section 4 we apply the Euler Top formalism in order to present solutions for spinning S^2 and S^3 branes with rotational symmetries $\prod_i SO(q_i)$.

In section 5 we examine the case of the spinning toroidal T^2 and T^3 branes including toroidal compactifications with rotational symmetries $\prod_i SU(q_i)$.

In the conclusions we interpret the solutions as nonperturbative type IIA-B solitons. Their energy is related non-perturbatively to the corresponding string coupling constants [15]. We close by discussing the relevance of our results to other recent work in the literature.

2. Lightcone equations of motion for p-branes and Nambu brackets

The Light Cone gauge of Nambu-Goto p-branes for flat space-times has been worked out in detail two decades ago [16]. The resulting Hamiltonian for the bosonic sector with zero flux background is given by:

$$H = \frac{T_p}{2} \int d^p \xi \sqrt{\gamma} [\dot{X}^{i^2} + \det[\partial_\alpha X^i \partial_\beta X^i]], \quad i = 1, \dots, D-2 \quad \alpha, \beta = 1, \dots, p \quad (2.1)$$

T_p is the brane tension, $d^p \xi \sqrt{\gamma}$ is the volume element in ξ -space

$$\partial_\alpha = \frac{\partial}{\partial \xi^\alpha} \quad \alpha = 1, \dots, p \quad (2.2)$$

It is easy to observe that the potential energy term of the Hamiltonian can be rewritten in terms of the Nambu p-bracket .

$$\det[\partial_\alpha X_i \partial_\beta X_i] = \frac{1}{p!} \sum_{i_1, \dots, i_p=1}^{D-2} \{X_{i_1}, \dots, X_{i_p}\}^2 \quad (2.3)$$

where

$$\{f_1, \dots, f_p\} \equiv \frac{1}{\sqrt{\gamma}} \epsilon^{\alpha_1 \dots \alpha_p} \partial_{\alpha_1} f_1 \dots \partial_{\alpha_p} f_p, \quad \alpha_1, \dots, \alpha_p = 1, \dots, p \quad (2.4)$$

The eqs of motion in terms of Nambu p-brackets read:

$$\ddot{X} = \frac{1}{(p-1)!} \{ \{X_i, X_{j_1}, \dots, X_{j_{D-1}}\}, X_{j_1}, \dots, X_{j_{p-1}} \} \quad (2.5)$$

$$i, j_1, \dots, j_{p-1} = 1, 2, \dots, D-2$$

The p-dimensional reparametrization invariance of the Lagrangian has been reduced after LC gauge fixing to p-volume preserving diffeomorphisms of the brane manifold M_p , $VolDiffs[M_p]$ [16]. This infinite dimensional gauge group contains elements not connected with the identity depending on the topology of M_p . The $VolDiffs[M_p]$ connected to the identity gauge transformations are generated by the constraints

$$\{\dot{X}_i, X_i\}_{\alpha, \beta} \equiv \frac{1}{V_{\alpha\beta}} (\partial_\alpha \dot{X}_i \partial_\beta \dot{X}_i - \partial_\beta \dot{X}_i \partial_\alpha X_i) = 0, \quad (2.6)$$

$$\alpha, \beta = 1, 2, \dots, p$$

where $V_{\alpha\beta}$ is the ξ_α, ξ_β part of the volume element $d^p \xi \sqrt{\gamma}$. The Nambu bracket is a generalization of the Poisson bracket of Classical Mechanics to "phase space" of any dimension p [17]. It is a completely antisymmetric multilinear function of f_1, \dots, f_p and satisfies two additional properties (α) Leibniz

$$\{f_1 \cdot g_1, f_2, \dots, f_p\} = f_1 \{g_1, f_2, \dots, f_p\} + g_1 \{f_1, \dots, f_p\} \quad (2.7)$$

and (β) the Fundamental Identity which generalizes the Jacobi identity. Furthermore, it generalizes Lie algebras and Poisson Manifolds to Nambu-Poisson and Nambu-Lie structures which turn out to be more rigid.

$$\begin{aligned} & \{ \{f_1, f_2, \dots, f_p\}, f_{p+1}, \dots, f_{2p-1} \} + \{ f_p, \{f_1, f_2, \dots, f_{p-1}, f_{p+1}\}, f_{p+2}, \dots, f_{2p-1} \} + \dots \\ & + \{ f_p, f_{p+1}, \dots, f_{2p-2}, \{f_1, f_2, \dots, f_{p-1}, f_{2p-1}\} \} = \{ f_1, f_2, \dots, f_{p-1}, \{f_p, f_{p+1}, \dots, f_{2p-1}\} \} \end{aligned} \quad (2.8)$$

There is one very interesting property of the Nambu bracket for spherical and toroidal p -branes. For S^p - p -dim. branes of spherical topology there is a natural system of functions e_1, \dots, e_{p+1} of the angles $\Omega = (\phi, \theta_1, \theta_2, \dots)$ where a unit vector in the direction Ω in $p + 1$ dimensional Euclidean space is expressed as

$$\hat{r} = (e_1, \dots, e_{p+1}) \quad (2.9)$$

with

$$e_1^2 + \dots + e_{p+1}^2 = 1 \quad (2.10)$$

These functions (polar coordinates of $p + 1$ -vectors) can be easily checked to satisfy

$$\{e_{i_1}, \dots, e_{i_p}\} = \epsilon_{i_1 \dots i_p i_{p+1}} e_{i_{p+1}}, \quad i_1, \dots, i_{p+1} = 1, \dots, p + 1 \quad (2.11)$$

For $p = 2$ they are

$$(e_1, e_2, e_3) = (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta) \quad (2.12)$$

Similarly for $p = 3$ we have

$$(e_1, e_2, e_3, e_4) = (\cos \phi \sin \theta_1 \sin \theta_2, \sin \phi \sin \theta_1 \sin \theta_2, \cos \theta_1 \sin \theta_2 \cos \theta_2) \quad (2.13)$$

The corresponding volume elements are:

$$p = 2 \quad d^2 \Omega = \sin \theta d\theta d\phi \quad (2.14)$$

and similarly

$$p = 3 \quad d^3 \Omega = \sin^2 \theta_2 \sin \theta_1 d\theta_1 d\theta_2 d\phi \quad (2.15)$$

The Poisson and Nambu brackets are defined correspondingly as

$$\{f_1, f_2\} \stackrel{p=2}{=} \frac{1}{\sin \theta} (\partial_\theta f_1 \partial_\phi f_2 - \partial_\phi f_1 \partial_\theta f_2) \quad (2.16)$$

and

$$\{f_1, f_2, f_3\} \stackrel{p=3}{=} \frac{1}{\sin^2 \theta_2 \sin \theta_1} \epsilon^{\alpha\beta\gamma} \partial_\alpha f_1 \partial_\beta f_2 \partial_\gamma f_3 \quad (2.17)$$

with $\alpha, \beta, \gamma = \theta_1, \theta_2, \phi$. For the torus T^p we have a flat measure for any p , $d\omega = d\sigma_1 \cdots d\sigma_p$ where

$$\sigma_\alpha \in (0, 2\pi), \quad \alpha = 1, \dots, p \quad (2.18)$$

the basis functions are

$$e_{\vec{n}} = e^{i\vec{n} \cdot \vec{\sigma}} \quad \vec{n} \in \mathcal{Z}^p \quad (2.19)$$

while their Nambu brackets are

$$\{e_{\vec{n}_1}, \dots, e_{\vec{n}_p}\} = i^p \det(\vec{n}_1, \dots, \vec{n}_p) \cdot e^{i(\vec{n}_1 + \dots + \vec{n}_p) \cdot \vec{\sigma}} \quad (2.20)$$

Volume preserving transformation can be defined through the Nambu bracket. For fixed f_1, \dots, f_{p-1} functions on the p -brane we define the generator

$$L_{(f_1, \dots, f_{p-1})} f = \{f_1, \dots, f_{p-1}, f\} \quad (2.21)$$

if f is functionally dependent on f_1, \dots, f_{p-1} the result is zero. The operation is restricted to satisfy fundamental identity (2.8). As an example for the 3-sphere S^3 for any two of the four functions e_1, e_2, e_3, e_4 the operator

$$L_{(e_i, e_j)} f = \{e_i, e_j, f\} \quad (2.22)$$

executes a rotation on the plane i, j and in general if $\alpha = \alpha_i \cdot e_i, \beta = \beta_j \cdot e_j$ with $(\alpha_i, \beta_j \in \mathcal{R})$,

$$L_{\alpha, \beta} f = \{\alpha, \beta, f\} \quad (2.23)$$

executes a rotation in the plane (α, β) . In a future work we shall present the structure of the algebras (2.11) for S^p, T^p .

The case $p = 2$ corresponds to the supermembrane and in this case there is a M(atrix) discretization by Goldstone, Gardner, Hoppe [18] which was revived in the late 80's [16] and late 90's as the M(atrix) model [10] proposal for M-theory. In the place of Poisson brackets one has commutators and in the place of target space $X_i(\xi_1, \xi_2, t), i = 1, \dots, D - 2$ of membrane coordinates one has $N \times N$ Hermitian matrices $A_i(t)$ (YM-mechanics in the Light Cone 10 + 1 dimensions). In 3 + 1 dimensions Yang-Mills mechanics was first studied by G.Savvidy [19]. The equations of motion and constraints are given by:

$$\ddot{A}_i = - [[A_i, A_j], A_i] \quad i, j = 1, \dots, D - 2 \quad (2.24)$$

and

$$[\dot{A}_i, A_i] = 0 \quad (2.25)$$

For the case of factorization of the time ansatz [12] it has been noticed that there is an isomorphism between the membrane $p = 2$ and the matrix model solutions. As a consequence any $p = 2$ spinning solution gives rise to a M(atrix) model solution. In the next section we will find the conditions for this type of motion by generalizing the Euler eqs for Rigid Body Motion of classical mechanics for p -branes in higher dimension.

3. P-brane Euler Tops in higher dimensions

In this part we derive the Euler eqs. for the purely rotational solutions of p-branes for any p. This type of motion presumably is the lowest in energy. Vibrational motion in radial or other directions costs more energy. Since p-branes possess elastic tension their equilibrium shape is controlled, for purely rotational motion, by the balance between the rotational forces and tension. We will specify the necessary and sufficient condition for this equilibrium ansatz.

The rotational or Euler Top motions of p-branes can be described by choosing some initial configuration $X_o^i(\xi)$, $\xi = (\xi_1, \dots, \xi_p)$, and

$$X^i(t) = R^{ij} X_o^j(\xi), \quad i, j = 1, \dots, D-2 \quad (3.1)$$

where R is a time dependent rotation matrix, $R \in SO(D-2)$ i.e. such that $R^T = R^{-1}$, $R(t=0) = I$ the $(D-2) \times (D-2)$ identity matrix. Let us introduce the moments of inertial tensor in the brane frame

$$I_B^{ik} = T_p \int d^p \xi \sqrt{\gamma} X_o^i(\xi) X_o^k(\xi), \quad i, k = 1, \dots, D-2 \quad (3.2)$$

and the angular momentum tensor which is conserved in the fixed space coordinate frame

$$L_S^{ij} = T_p \int d^p \xi \sqrt{\gamma} (\dot{X}^i X^j - \dot{X}^j X^i), \quad i, j = 1, \dots, D-2 \quad (3.3)$$

The two frames, Brane and Space, are connected through the Rotation Matrix R. We introduce the angular momentum in the Brane frame L_B and the Moment of Inertia in the Space Frame I_S

$$I_S = R(t) \cdot I_B \cdot R^{-1}(t) \quad (3.4)$$

and

$$L_B = R^{-1}(t) \cdot L_S \cdot R(t) \quad (3.5)$$

The linking quantity between the angular momentum L and the moment of inertia tensor I is of course the angular velocity matrix in the two frames:

$$\omega_S = \dot{R} R^{-1} \quad (3.6)$$

and

$$\omega_B = R^{-1} \dot{R} \quad (3.7)$$

From the above definitions we obtain

$$L_B = \omega_B I_B + I_B \omega_B \quad (3.8)$$

and

$$L_S = \omega_S I_S + I_S \omega_S = R \cdot L_B \cdot R^{-1} \quad (3.9)$$

From the conservation of L_S we obtain

$$\dot{L}_B + [\omega_B, L_B] = 0 \quad (3.10)$$

and from (3.8) the Euler eqs [20]

$$\dot{\omega}_B I_B + I_B \dot{\omega}_B + [\omega_B^2, I_B] = 0 \tag{3.11}$$

The above equation discloses the richness of rigid body dynamics generalized to higher dimensions [20]. For the p-branes rotational motion we make the ansatz (3.1). The constraints impose the condition on $\omega_B(t=0) = \omega_{B_o}$

$$\omega_{B_o}^{ij} \{X_o^i, X_o^j\}_{\xi_\alpha, \xi_\beta} = 0, \quad \alpha, \beta = 1, \dots, p \tag{3.12}$$

This condition is easily satisfied if we partition the i,j range into a direct sum structure $(i_1, j_1), (i_2, j_2), \dots$

$$\omega_{B_o} = \omega_{B_o}^1 \oplus \omega_{B_o}^2 \oplus \dots \tag{3.13}$$

and impose $\{X_o^{iq}, X_o^{jq}\}_{\xi_\alpha, \xi_\beta} = 0, \quad \forall q = 1, 2, \dots$. This is the general structure of our ansatz in the next sections for S^2, S^3, T^2, T^3 . On the other hand the eqs. of motion (2.5) produce the following additional constraints:

$$v^{ij} X_o^j = \frac{1}{(p-1)!} \{ \{X_o^i, X_o^{k_1}, \dots, X_o^{k_{p-1}}\}, X_o^{k_1}, \dots, X_o^{k_{p-1}} \} \tag{3.14}$$

$$i, k_1, \dots, k_{p-1} = 1, \dots, D-2$$

where for all times

$$v^{ij} \equiv (R^{-1} \ddot{R})^{ij} \tag{3.15}$$

and X_o^i should close the algebra (3.14). In what follows, we are going to see that this is guaranteed for special functions X_o^i . The implication of rel.(3.15) is that

$$\ddot{R} = R \cdot v \tag{3.16}$$

with $R(t=0) = I, R^T R = I$ and v is constant. The only solution to these requirements is

$$R(t) = e^{\Omega \cdot t}, \quad \Omega^T = -\Omega \tag{3.17}$$

and thus v is a symmetric non-negative definite matrix

$$v = \Omega^2 \tag{3.18}$$

The energetics of this ansatz goes as follows. The energy of the configuration

$$E = \frac{T_p}{2} \int d^p \xi \sqrt{\gamma} \left[\dot{R}^{ij} X_o^j X_o^k \dot{R}^{ki} + \frac{1}{p!} \{X_o^{i_1}, \dots, X_o^{i_p}\}^2 \right] \tag{3.19}$$

consists of two conserved pieces: The potential energy V

$$V = \frac{T_p}{2p!} \int d^p \xi \sqrt{\gamma} \{X_o^{i_1}, \dots, X_o^{i_p}\}^2 \tag{3.20}$$

and the kinetic energy which is expressed in terms of the conserved angular momentum (L_S is minus the usual angular momentum)

$$E_{\text{kin}} = -\frac{1}{2} \text{tr} \omega_S \cdot I_S \cdot \omega_S = -\frac{1}{4} \text{tr} L_S \cdot \omega_S \tag{3.21}$$

By integrating the equation of equilibrium of forces after multiplying by X_o^i eq. (3.14) we get for the potential energy:

$$T_p \int d^p \xi \sqrt{\gamma} v^{ij} X_o^i X_o^j = -\frac{T_p}{(p-1)!} \int d^p \xi \sqrt{\gamma} \{X_o^{i_1}, \dots, X_o^{i_p}\}^2 = -2pV \quad (3.22)$$

or

$$tr v \cdot I_B = -2pV \quad (3.23)$$

From (3.6)–(3.7) we obtain (virial theorem)

$$\omega_B = \omega_S = \Omega \quad (3.24)$$

and thus

$$V = -\frac{1}{2p} tr \Omega^2 I_B \quad (3.25)$$

$$E_{\text{kin}} = -\frac{1}{2} tr \Omega^2 I_B = pV \quad (3.26)$$

and

$$E_{\text{tot}} = -\left(\frac{1}{2} + \frac{1}{2p}\right) tr \Omega^2 I_B \quad (3.27)$$

Finally the relation of E_{tot} to the conserved angular momenta is

$$E_{\text{tot}} = -\frac{1}{4} \left(1 + \frac{2}{p}\right) tr \Omega L_S \quad (3.28)$$

4. Spherical p-brane Tops (S^2 , S^3)

4.1 S^2 Tops

In this section we exhibit new spinning $p = 2$ and $p = 3$ spherical brane solutions with rotational symmetries $\prod_i SO(q_i)$. We render transparent the role of the symmetry algebras which are formed by the Nambu-Poisson brackets and clarify the minimum energy character of the p-Euler Tops. For S^2 ($p = 2$) the relevant $SO(3)$ algebra for the basis functions

$$\begin{aligned} (e_1, e_2, e_3) &= (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta) \\ \{e_i, e_j\} &= -\epsilon_{ijk} e_k \end{aligned} \quad (4.1)$$

is responsible for the polynomially generated universal enveloping algebra, the $\text{SDiff}(S^2)$. It is known that the only finite dimensional subalgebras of $\text{SDiff}(S^2)$ is $SO(3)$. Thus factorization with a finite number of time dependent modes can be found by using only the e_i s. We propose a generalization of embeddings for S^2 in $9 - \text{dim}$.

$$R^9 = R^{q_1} \times R^{q_2} \times R^{q_3}, \quad q_1 + q_2 + q_3 = 9 \quad (4.2)$$

as follows:

$$\begin{aligned} X_i &\equiv x_i(t) \cdot e_1 \\ Y_j &\equiv X_{q_1+j} = y_j \cdot e_2 \\ Z_k &\equiv X_{q_1+q_2+k} = z_k \cdot e_3 \end{aligned} \quad (4.3)$$

where $(i, j, k = 1, \dots, q_1, q_2, q_3)$ respectively with $q_1 + q_2 + q_3 = 9$ and the q_i s are nonzero integers. The case $q_1 = q_2 = q_3 = 2$ for the matrix model has been studied in ref. [21, 22] while for the membrane in ref [12, 22]. In principle one of the q_i , $i = 1, 2, 3$ may be zero. The constraints

$$\sum_{i=1}^9 \{\dot{X}_i, X_i\} = 0 \tag{4.4}$$

are automatically satisfied.

The functions x_i, y_j, z_k functions which determine the simultaneous time evolution of every point of S^2 in R^9 satisfy the eqs. of motion

$$\ddot{\vec{x}} = -\vec{x}(r_y^2 + r_z^2) \tag{4.5}$$

By cyclic permutation on the x, y, z one obtains similarly the eqs for \vec{y} and \vec{z} with $\vec{x} = (x_1, \dots, x_{q_1})$, $\vec{y} = (y_1, \dots, y_{q_2})$, $\vec{z} = (z_1, \dots, z_{q_3})$ and

$$r_x^2 = \sum_{i=1}^{q_1} x_i^2, \quad r_y^2 = \sum_{j=1}^{q_2} y_j^2, \quad r_z^2 = \sum_{k=1}^{q_3} z_k^2 \tag{4.6}$$

We see that eqs.(4.3) admit an $SO(q_1) \times SO(q_2) \times SO(q_3) \subset SO(9)$ rotational symmetry. The Hamiltonian of the ansatz

$$H = \frac{T_2}{2} \int_{S^2} d^2\xi \left[\dot{X}_i^2 + \frac{1}{2} \{X_i, X_j\}^2 \right] \tag{4.7}$$

can be calculated by the use of the orthogonality relation

$$\int_{S^2} d^2\xi e_k \cdot e_l = \frac{4\pi}{3} \delta_{k,l}, \quad k, l = 1, 2, 3 \tag{4.8}$$

We find

$$E = \frac{2\pi}{3} T_2 \left[\dot{\vec{x}}^2 + \dot{\vec{y}}^2 + \dot{\vec{z}}^2 + r_x^2 r_y^2 + r_x^2 r_z^2 + r_y^2 r_z^2 \right] \tag{4.9}$$

In order to relate the Energy with $SO(d_1), SO(d_2), SO(d_3)$ angular momenta we observe that for each component separately we have

$$(L_z)_{mn} = \frac{4\pi T_2}{3} (l_z)_{m,n}, \quad m, n = 1, \dots, q_3 \tag{4.10}$$

The same will hold true for $(L_y)_{kl}$ and $(L_x)_{ij}$ with $k, l = 1, \dots, q_2$ and $i, j = 1, \dots, q_1$ respectively.

Here l_x, l_y, l_z are given by

$$(l_x)_{ij} = \dot{x}_i x_j - \dot{x}_j x_i \tag{4.11}$$

Similarly for $(l_y)_{kl}$ and $(l_z)_{mn}$.

The higher dimensional kinetic terms $\dot{\vec{x}}^2, \dots$ can be expressed in terms of the radial and angular variables as:

$$\dot{\vec{x}}^2 = \dot{r}_x^2 + \frac{l_x^2}{r_x^2} \tag{4.12}$$

Then the energy is given in terms of l_x, l_y, l_z and r_x, r_y, r_z as:

$$E = \frac{2\pi T_2}{3} (E_{\text{kin}} + V_{\text{eff}}) \quad (4.13)$$

where

$$\begin{aligned} E_{\text{kin}} &= \dot{r}_x^2 + \dot{r}_y^2 + \dot{r}_z^2 \\ V_{\text{eff}} &= \frac{l_x^2}{r_x^2} + \frac{l_y^2}{r_y^2} + \frac{l_z^2}{r_z^2} + r_x^2 r_y^2 + r_x^2 r_z^2 + r_y^2 r_z^2 \end{aligned} \quad (4.14)$$

We are now in the position to make the connection between this ansatz and the Euler-Top formalism of ch.3. Due to the breaking of rotational symmetry $SO(9)$ to $SO(q_1) \times SO(q_2) \times SO(q_3)$ the time-evolution of the vector $\vec{x}(t), \vec{y}(t), \vec{z}(t)$ is described by:

$$\begin{aligned} \vec{x}(t) &= e^{\Omega_x \cdot t} \vec{x}_o \\ \vec{y}(t) &= e^{\Omega_y \cdot t} \vec{y}_o \\ \vec{z}(t) &= e^{\Omega_z \cdot t} \vec{z}_o \end{aligned} \quad (4.15)$$

By using $SO(q_1), SO(q_2), SO(q_3)$ rotations we can bring the vectors $\vec{x}_o, \vec{y}_o, \vec{z}_o$ to their corresponding first axes:

$$\begin{aligned} \vec{x}_o &= R_x(1, \dots, 0) \quad q_1 - \text{components} \\ \vec{y}_o &= R_y(1, \dots, 0) \quad q_2 - \text{components} \\ \vec{z}_o &= R_z(1, \dots, 0) \quad q_3 - \text{components} \end{aligned} \quad (4.16)$$

By keeping the position vectors fixed we can bring the initial velocities to the planes $x^1 x^2, y^1 y^2, z^1 z^2$. Thus each $\Omega_i (i = x, y, z)$ angular velocity matrix becomes

$$\Omega_i = \begin{pmatrix} 0 & -\omega_i \\ \omega_i & 0 \end{pmatrix} \quad (4.17)$$

in their respective planes and zero for all others. The moment of inertia tensor acquires a similar form:

$$\begin{aligned} I_B &= I_x \oplus I_y \oplus I_z \\ I_i &= \frac{2\pi T_2}{3} \begin{pmatrix} R_i^2 & 0 \\ 0 & 0 \end{pmatrix}, \quad i = x, y, z \end{aligned} \quad (4.18)$$

and so $L_B = L_S$ where

$$\begin{aligned} L_B &= \omega_x I_x \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \oplus \omega_y I_y \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \oplus \omega_z I_z \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ &\equiv L_x \oplus L_y \oplus L_z \end{aligned} \quad (4.19)$$

The total energy according to rel. (3.28) is:

$$E = \frac{1}{2} (\omega_x^2 I_x + \omega_y^2 I_y + \omega_z^2 I_z) \quad (4.20)$$

The balance of force condition (or eq. of motion) relates the angular momenta with the radii of rotation as

$$\omega_x^2 = R_y^2 + R_z^2 \quad (4.21)$$

Similarly for ω_y and ω_z .

These equations are identical to the ones obtained from the minimization of the effective potential V_{eff} (4.10) which lead to constant radii solutions:

$$r_i = R_i, \quad i = x, y, z \quad (4.22)$$

We now proceed to present details of the solutions of the minimization conditions which provide an interesting complex dependence of the Energy (4.20) on the angular momenta L_x, L_y, L_z . The extrema of the Energy are given by constant in time radii r_x, r_y, r_z satisfying:

$$\frac{\partial V_{\text{eff}}}{\partial r_x} = -\frac{2l_x^2}{r_x^3} + 2r_x(r_y^2 + r_z^2) = 0 \quad (4.23)$$

and so on for r_y, r_z . The system of equations obtained are

$$r_x^4(r_y^2 + r_z^2) = l_x^2 \quad (4.24)$$

By permutation symmetry $x \leftrightarrow y, l_x \leftrightarrow l_y, \dots$ etc one gets the rest which can be solved for general l_x^2, l_y^2, l_z^2 .

We exhibit solutions only for the two simplest cases:

$$l_i^2 \equiv l^2, \quad r_i^2 \equiv r^2, \quad i = x, y, z \quad (4.25)$$

the completely symmetric case (S) and

$$l_x^2 = l_y^2 = l^2 \neq l_z^2, \quad r_x^2 = r_y^2 = r^2 \neq r_z^2 \quad (4.26)$$

the axially symmetric case (A). Before that though we will demonstrate that the extrema (4.22) are local minima of the energy. Indeed, by taking the second variation of the potential at the extrema [12, 22];

$$\frac{\partial^2 V}{\partial r_i \partial r_j} \Big|_{i,j=x,y,z} = 4 \begin{pmatrix} 2(r_y^2 + r_z^2) & r_x r_y & r_x r_z \\ r_y r_x & 2(r_x^2 + r_z^2) & r_y r_z \\ r_z r_x & r_z r_y & 2(r_x^2 + r_y^2) \end{pmatrix} \quad (4.27)$$

we check that this is a real symmetric matrix (real eigenvalues) but also positive definite (positive eigenvalues) i.e. for arbitrary real vectors $\xi_i \in \mathcal{R}$, $i = x, y, z$ we find

$$\xi_i \xi_j \frac{\partial^2 V}{\partial r_i \partial r_j} > 0 \quad (4.28)$$

We shall compare now energetically the symmetric with the axially symmetric cases (4.25)–(4.26). For the symmetric case (S) we find

$$r_S^2 \equiv r^2 = \left(\frac{l^2}{2}\right)^{1/3}, \quad V_{\text{eff}}^{\text{min}} = V_S = \frac{9}{4^{1/3}} \cdot (l^2)^{2/3} \quad (4.29)$$

For the axially symmetric case we find

$$\begin{aligned}
 r_z^2 &= \frac{l_z}{2l^{2/3}} \left(l_z + \sqrt{l_z^2 + 8l^2} \right)^{1/3} \\
 r_\alpha^2 &= \frac{2l^{4/3}}{\left(l_z + \sqrt{l_z^2 + 8l^2} \right)^{2/3}} \\
 V_{\text{eff}}^{\text{min}} \equiv V_\alpha &= \frac{6l^{2/3}}{\left(l_z + \sqrt{l_z^2 + 8l^2} \right)^{4/3}} \left[l_z(l_z + \sqrt{l_z^2 + 8l^2}) + 2l^2 \right]
 \end{aligned}
 \tag{4.30}$$

In order to compare the two minima we rescale $l_z = \lambda l$ and we identify it (1) in each of the two cases . We find for the ratio

$$\frac{V_\alpha}{V_s} = f(\lambda) = \frac{2^{5/3}}{3} \frac{\lambda \left(\lambda + \sqrt{\lambda^2 + 8} \right) + 2}{\left(\lambda + \sqrt{\lambda^2 + 8} \right)^{4/3}}
 \tag{4.31}$$

while for the radii: $r_s = \left(\frac{l^2}{2} \right)^{1/3}$

$$\begin{aligned}
 \frac{r_z^2}{r_s^2} &= \frac{\lambda}{2^{2/3}} \left(\lambda + \sqrt{\lambda^2 + 8} \right)^{1/3} \\
 \frac{r_\alpha^2}{r_s^2} &= \frac{2^{4/3}}{\left(\lambda + \sqrt{\lambda^2 + 8} \right)^{2/3}}
 \end{aligned}
 \tag{4.32}$$

where for $\lambda \rightarrow 1$, $f(\lambda) \rightarrow 1$, and $r_z^2/r_s^2 = r_\alpha^2/r_s^2 = 1$.

From the above analysis we deduce that if the membrane length in one dimensionality (say q_3) is much bigger than the other two (q_1, q_2) it loses energy with respect to the symmetric case, while if it is much smaller than the other two it gains energy. The expressions of the Energy as a function of the angular momenta and tension shows the non-perturbative character of the spinning solutions. It also affords us the possibility to quantize the rotational modes of the S^2 membrane by using L^2 and L_z^2 as Casimirs (with eigenvalues $\hbar n(n+q-2)$, $n = 0, 1, 2, \dots$ for $SO(q)$ of the $SO(q_1), SO(q_2), SO(q_3)$ rotational groups). The classical S^2 spinning membranes live in a 6-dims out of the total nine while the quantum one occupies all dimensions due to the rotational wave functions $SO(q_1), SO(q_2), SO(q_3)$ (spherical harmonics) in $q_1 + q_2 + q_3 = 9$ dimensions. Concerning the stability of our solution, as we have already shown, there is classical and quantum mechanical perturbative stability for the radial modes and quadratic expansion in r_x, r_y, r_z around the minima will exhibit the perturbative vibrational spectrum. Stability for the multipole in θ, ϕ fluctuations exists only for the symmetric case $l_x = l_y = l_z$ as can be shown by using the results of [12, 22]. The geometry of the ansatz with rotating axes is that of an ellipsoid which at any time satisfies the eqs:

$$\frac{\vec{X}^2}{r_x^2} + \frac{\vec{Y}^2}{r_y^2} + \frac{\vec{Z}^2}{r_z^2} = 1.
 \tag{4.33}$$

On the other hand by suitable rotations only three planes survive, i.e. (12) of the q_1 , q_2 and q_3 dimensions respectively. Thus the two dimensional S^2 surface is moving in a fixed 5-dimensional ellipsoid in the 9-dim space. We can use this observation to argue for M-theory curved gravitational backgrounds with 5-dimensional ellipsoidal minimal sub-manifolds (pp waves for example). Our spinning solutions are isometrically embedable in these backgrounds, i.e. they satisfy eqs. of motion in these backgrounds.

4.2 S^3 Tops

We close this section by presenting new spinning S^3 -brane solutions. The branes for $p = 3$ attract a lot of attention due to their possible role as fundamental particles, YM-Gravity dualities [1, 11], Matrix Cosmology [6, 7], giant gravitons [1, 4] etc. Although pp-waves with fluxes present interesting backgrounds, we will hereby consider only flat LC-spacetimes in order to show that local minima of the energy can be found by appropriately balancing, generalizing spinning solutions, rotation with tension forces in this case too. The Hamiltonian for an S^3 brane (see ch.2) in LC gauge can be written in terms of the Nambu 3-brackets

$$H = \frac{T_3}{2} \int d\Omega_3 \left[\dot{X}^i{}^2 + \frac{1}{3!} \{X^i, X^j, X^k\}^2 \right] \quad (4.34)$$

so that the resulting equations of motion and constraints are:

$$\begin{aligned} \ddot{X}^i &= \frac{1}{2} \{ \{X^i, X^j, X^k\}, X^j, X^k \}, \quad i, j, k = 1, \dots, d \leq D - 2 \\ \{ \dot{X}^i, X^i \}_{\xi_\alpha, \xi_\beta} &= 0, \quad \alpha \neq \beta = \theta, \phi, \psi \end{aligned} \quad (4.35)$$

with

$$\begin{aligned} d\Omega_3 &= \sin^2 \psi \sin \theta d\psi d\theta d\phi \\ (\xi_\alpha) &= (\theta, \phi, \psi), \quad 0 \leq \theta, \psi \leq \pi, 0 \leq \phi \leq 2\pi \end{aligned} \quad (4.36)$$

and the Nambu 3-bracket for S^3 :

$$\{X^i, X^j, X^k\} = \frac{-1}{\sin^2 \psi \sin \theta} \cdot \epsilon^{\alpha\beta\gamma} \partial_\alpha X^i \partial_\beta X^j \partial_\gamma X^k, \quad \xi_1 = \theta, \xi_2 = \phi, \xi_3 = \psi \quad (4.37)$$

As we discussed in ch.2 for S^3 (here $p = 3$) there are $p+1 = 4$ functions ($e_1^2 + e_2^2 + e_3^2 + e_4^2 = 1$)

$$\begin{aligned} e_1 &= \cos \phi \sin \theta \sin \psi \\ e_2 &= \sin \phi \sin \theta \sin \psi \\ e_3 &= \cos \theta \sin \psi \\ e_4 &= \cos \psi \end{aligned} \quad (4.38)$$

closing the Nambu-bracketed (volume preserving S^3 , Diff's) algebra, here global $SO(4)$ rotations

$$\{e_\alpha, e_\beta, e_c\} = -\epsilon_{\alpha\beta cd} e_d, \quad \alpha, \beta, c, d = 1, 2, 3, 4 \quad (4.39)$$

As is the case with S^2 this is crucial for the factorization of time and θ, ϕ, ψ dependence of the eqs. of motion. Thus with an analogous to S^2 ansatz satisfying the constraints (4.43)

$$\begin{aligned}
 X^i &= x^i(t)e_1, \quad i = 1, \dots, q_1 \\
 Y^j &= X^{j+q_1} = y^j(t)e_2, \quad j = 1, \dots, q_2 \\
 Z^k &= X^{k+q_1+d_2} = z^k(t)e_3, \quad k = 1, \dots, q_3 \\
 W^l &= X^{l+q_1+q_2+q_3} = w^l(t)e_4, \quad l = 1, \dots, q_4
 \end{aligned} \tag{4.40}$$

with $q_1 + q_2 + q_3 + q_4 = d \leq D - 2$, $q_\alpha \geq 0$, $\alpha = 1, 2, 3, 4$ we obtain:

$$\ddot{\vec{x}} = -\vec{x}(r_x^2 r_z^2 + r_y^2 r_w^2 + r_z^2 r_w^2) \tag{4.41}$$

By cyclic permutation one obtains similarly the eqs of motion for $\vec{y}, \vec{z}, \vec{w}$. where r_x, r_y, r_z, r_w are the lengths of the vectors $\vec{x}, \vec{y}, \vec{z}, \vec{w}$ respectively. From 4.40 we see that the rotational symmetry $SO(d)$ is broken down to $SO(q_1) \times SO(q_2) \times SO(q_3) \times SO(q_4)$. Of course we must have $q_{\text{alpha}} \geq 2$ in order to have at least $SO(2)$ rotational symmetry. Otherwise (i.e. if some $q_\alpha = 1$) we have less rotational symmetry. The Energy-Angular momenta of the ansatz are:

$$\begin{aligned}
 E &= \frac{T_3 \text{Vol}(S^3)}{2} \frac{1}{4} [\dot{r}_x^2 + \dot{r}_y^2 + \dot{r}_z^2 + \dot{r}_w^2 + \frac{l_x^2}{r_x^2} + \frac{l_y^2}{r_y^2} + \frac{l_z^2}{r_z^2} + \frac{l_w^2}{r_w^2} \\
 &\quad + r_x^2 r_y^2 r_z^2 + r_x^2 r_y^2 r_w^2 + r_y^2 r_z^2 r_w^2 + r_x^2 r_z^2 r_w^2]
 \end{aligned} \tag{4.42}$$

where $\text{Vol}(S^3) = 2\pi^2$ and the angular momenta are

$$L_i^2 = \left(\frac{T_3 \pi^2}{2 \cdot 2} \right)^2 l_i^2, \quad i = x, y, z, w \tag{4.43}$$

and

$$l_x^2 = \sum_{i \neq j=1}^{q_1} (\dot{x}_i x_j - \dot{x}_j x_i)^2 \tag{4.44}$$

Similarly for l_y, l_z, l_w . If all $l_{x,y,z,w}^2$ are different from zero the minimization condition for the V_{eff} is equivalent to constant radii solutions

$$l_x^2 = r_x^4 (r_y^2 r_z^2 + r_y^2 r_w^2 + r_z^2 r_w^2) \tag{4.45}$$

Indeed, we can check that these are local minima ($\frac{\partial^2 V}{\partial r_\alpha \partial r_\beta} |_{\text{minima}}$ is positive definite). With analogous arguments with the S^2 case the minimization condition can be solved due to permutation symmetry ($x \rightarrow y \rightarrow z \rightarrow w$) with in general fourth order polynomial equations. Here we exhibit the two simplest cases: (a) symmetric, $r_x = r_y = r_z = r_w = R$, $l_x = l_y = l_z = l_w = l$, and (b) axially symmetric $r_x = r_y = r_z = R$, $r_w = R_w$, and $l_x = l_y = l_z = l, l_w$. For the symmetric case we get:

$$R_{\text{sym}}^2 = \left(\frac{l^2}{3} \right)^{1/4} \tag{4.46}$$

$$E_{\text{sym}} = 2T_3 \text{Vol}(S^3) \left(\frac{l^2}{3} \right)^{3/4} \tag{4.47}$$

For the axisymmetric case the radii are:

$$R^2 = \left(\frac{l_w^2}{3}\right)^{1/4} \left[\sqrt{1 + 3\frac{l^2}{l_w^2}} - 1 \right]^{1/2}$$

$$R_w^2 = \frac{(l_w^2/3)^{1/4}}{[\sqrt{1 + 3l^2/l_w^2} - 1]^{1/2}} \tag{4.48}$$

and the energy

$$E_{ax} = \frac{T_3 \text{Vol}(S^3)}{2} \left(\frac{l_w^2}{3}\right)^{3/4} \left[2 + \sqrt{1 + 3\frac{l^2}{l_w^2}} \right] \left[\sqrt{1 + 3\frac{l^2}{l_w^2}} - 1 \right]^{1/2} \tag{4.49}$$

By rescaling $l_w^2 = \lambda l^2$ we find

$$\frac{E_{ax}}{E_{\text{sym}}} = \frac{\lambda^{3/4}}{4} \left(2 + \sqrt{1 + \frac{3}{\lambda}} \right) \left(-1 + \sqrt{1 + \frac{3}{\lambda}} \right)^{1/2} \tag{4.50}$$

also

$$\frac{R_w^2}{R^2} = \frac{1}{\sqrt{1 + \frac{3}{\lambda}} - 1} \tag{4.51}$$

For $\lambda = 1$ we have the symmetric case. For $\lambda \rightarrow 0$, we find qualitatively similar results with S^2 : For $\lambda \rightarrow \infty$ we find

$$\frac{E_{ax}}{E_s} \xrightarrow{\lambda \rightarrow 0} \frac{3^{3/4}}{4} < 1$$

$$\frac{E_{ax}}{E_s} \xrightarrow{\lambda \rightarrow \infty} \lambda^{1/4} \frac{3}{2^2} \left(\frac{3}{2}\right)^{1/2} > 1 \tag{4.52}$$

As far as the time dependence is concerned we can choose without loss of generality 4-planes $x^1x^2, y^1y^2, z^1z^2, w^1w^2$ where the initial position and velocity vectors belong. Then the ansatz (4.47) of constant radii (at the minima) $r_x = R_x, r_y = R_y, r_z = R_z, r_w = R_z$

$$\vec{\dot{x}}(t) = e^{\Omega_x t} \vec{x}(0) \tag{4.53}$$

Similarly for $\vec{\dot{y}}, \vec{\dot{z}}, \vec{\dot{w}}$.

With $\Omega_i = \begin{pmatrix} 0 & -w_i \\ w_i & 0 \end{pmatrix}$, $i = x, y, z, w$ and the balancing of force conditions give (see ch.3) $v = \Omega_x^2 \oplus \Omega_y^2 \oplus \Omega_z^2 \oplus \Omega_w^2$ with

$$\omega_x^2 = R_x^2 R_y^2 + R_z^2 R_w^2 + R_y^2 R_w^2 \tag{4.54}$$

By cyclic permutation of the indices one obtains the other components as well.

These relations are identical to the minimization conditions (4.45) since $l_i = \omega_i R_i^2, i = x, y, z, w$. As a result given the constants of motion l_x, l_y, l_z, l_w , the R's are determined. The stability of the spinning S^3 -brane solutions has been shown only for the radial modes.

For the symmetric case, (all l's, R's are equal), we conjecture that we have full stability i.e. by including perturbations of general S^3 multipole-vibrational modes.

It is possible to choose the dimension of the ansatz $d = q_1 + q_2 + q_3 + q_4 < D - 2$, D the $p = 3$ critical dimension, i.e. $D = 6, 8$ and for the rest $D - 2 - d$ we select constant values for the coordinates $X^i, i = D - 2 - d, D - 1 - d, \dots, D - 2$. If $D - 2 - d = 3$ our physical space, then we have S^3 -particles with Kaluza-Klein charges- (internal angular momenta, as is also the case with S^2). The QM of the rotational modes plus quadratic vibrational ones can be carried out by using only algebraic functions of $SO(q_i)$ Casimirs.

5. Toroidal p-brane Tops (T^2, T^3) on $C^k \times T^m$

5.1 T^2 spinning Tops

In this section we propose some new spinning toroidal p-brane solutions with some of the higher dimensions compactified in Toroidal spaces. Double dimensional reduction of the $p = 2$ Toroidal Supermembrane leads to type IIA string theory. With the addition of an S^1 compactification followed by T-duality a connection is made with Type IIB string Theory . In order to proceed we choose $d < D - 2$ dimensions to be an even number $d = 2k$. We collect the coordinates $X^1, X^2, \dots, X^{2k-1}, X^{2k}$ into complex pairs,

$$Z^i = X^{2i-1} + iX^{2i}, \quad i = 1, \dots, d/2 \tag{5.1}$$

We identify the rest ones $D - 2 - d = m$ as Y^a with $a = 1, \dots, m$. The Hamiltonian can be identified from ch.2 to be

$$H = \frac{T_p}{2} \int d^p \xi [\dot{X}^i{}^2 + \det g_{\alpha\beta}] \tag{5.2}$$

where $g_{\alpha\beta} = \partial_\alpha X^i \partial_\beta X^i$, $(\alpha, \beta = 1, \dots, p)$ is the induced metric. The connection with the Nambu Poisson bracket is established through the identity:

$$\det g_{\alpha\beta} = \frac{1}{p!} \epsilon_{\alpha_1 \dots \alpha_p} \epsilon_{\beta_1 \dots \beta_p} g_{\alpha_1, \beta_1} \dots g_{\alpha_p, \beta_p}, \quad \alpha_1(\beta_1), \dots, \alpha_p(\beta_p) = 1, \dots, p \tag{5.3}$$

for the case $p = 2$ by taking into account the pairing eq.(5.1) we find

$$g_{\alpha\beta} = \frac{1}{2} (\partial_\alpha Z^i \partial_\beta \bar{Z}^i + \partial_\alpha \bar{Z}^i \partial_\beta Z^i) + \partial_\alpha Y^a \partial_\beta Y^a, \quad i = 1, \dots, k \quad a = 1, \dots, m \tag{5.4}$$

By applying rel.(5.3) to the case of $p = 2$ Torus T^2 the Hamiltonian becomes

$$H = \frac{T_p}{2} \int d^2 \sigma [|\dot{Z}^i|^2 + |\dot{Y}^a|^2 + \frac{1}{4} |\{Z^i, Z^j\}|^2 + \frac{1}{4} |\{Z^i, \bar{Z}^j\}|^2 + |\{Z^i, Y^a\}|^2 + \frac{1}{2} |\{Y^a, Y^b\}|^2] \tag{5.5}$$

$i, j = 1, \dots, k \quad a, b = 1, \dots, m \quad \vec{\sigma} = (\sigma_1, \sigma_2) \in (0, 2\pi)^2$

The constraints become:

$$\{\dot{Z}^i, \bar{Z}^i\} + c.c. + \{\dot{Y}^a, Y^a\} = 0 \tag{5.6}$$

The eqs of motion for the Hamiltonian (5.5) are:

$$\begin{aligned}\ddot{Z}^i &= \frac{1}{2}\{\{Z^i, Z^j\}, \bar{Z}^j\} + \frac{1}{2}\{\{Z^i, \bar{Z}^j\}, Z^j\} + \frac{1}{2}\{\{Z^i, Y^a\}, Y^a\} \\ \ddot{Y}^a &= \frac{1}{2}\{\{Y^a, Z^i\}, \bar{Z}^i\} + \frac{1}{2}\{\{Y^a, \bar{Z}^i\}, Z^i\} + \frac{1}{2}\{\{Y^a, Y^b\}, Y^b\} \\ i, j &= 1, \dots, k \quad a, b = 1, \dots, m\end{aligned}\tag{5.7}$$

Before we proceed with the factorization ansatz let us demonstrate that the Hamiltonian (5.5) along with the eqs.(5.7) with a suitable dimensional reduction (double or multiple) describe LC gauge fixed closed string theory (the Bosonic part). Choose all the Y^a compactified on a torus T^m , $a = 1, \dots, m$ with radii R_a .

$$Y^a = R_a \cdot \vec{m}_a \cdot \vec{\xi} + \frac{2\pi k^a}{R_a} \cdot t\tag{5.8}$$

$\vec{m}_a = (m_a^1, m_a^2) \in Z^2$ the windings and k^a the KK integer momenta. We also assume that all the $Z^i, i = 1, \dots, k$ depend only on ξ_1 . For the reduced Hamiltonian we get

$$H_{\text{red}} = \pi T_2 \int d\xi_1 [|\dot{Z}^i|^2 + k|\partial_{\sigma_1} Z^j|^2]\tag{5.9}$$

with $k = \sum_a R_a^2 (m_a^1)^2$. By rescaling the time $t = \frac{1}{\sqrt{k}}\tau$ and by calling $\xi_1 = \xi$ we obtain

$$H_{\text{string}} = \frac{T_1}{2} \int_0^{2\pi} d\xi_1 [|\partial_\tau Z^i|^2 + |\partial_\xi Z^i|^2]\tag{5.10}$$

where $T_1 = 2\pi k T_2$. We will consider special embeddings of the T^2 in $C^k \times T^m$, toroidally compactified.

$$\begin{aligned}Z^i &= \zeta^i(t) e^{i\vec{n}_i \cdot \vec{\xi}}, \quad i = 1, \dots, k \\ Y^a &= R_a \cdot \vec{m}_a \cdot \vec{\xi} + \frac{2\pi k^a}{R_a} \cdot t, \quad a = 1, \dots, m\end{aligned}\tag{5.11}$$

R_a are the radii of T^m and $\vec{m}_a = (m_a^1, m_a^2) \in Z^2$ are the winding numbers and k^a the KK momenta. It is trivial to see that the eqs. of motion for Y^a as well as the constraints are automatically satisfied. As for the Hamiltonian we find

$$H = 2\pi^2 T_2 \left[\sum_i \left(|\dot{\zeta}^i|^2 + k_i |\zeta^i|^2 \right) + \frac{1}{2} \sum_{i,j} \nu_{ij} |\zeta^i|^2 |\zeta^j|^2 \right], \quad i = 1, \dots, k\tag{5.12}$$

where

$$\nu_{ij} = (\vec{n}_i \times \vec{n}_j)^2, \quad k_i = \sum_a R_a^2 (\vec{m}_a \times \vec{n}_i)^2\tag{5.13}$$

and $(\vec{n} \times \vec{m}) = n_1 m_2 - n_2 m_1$. The eqs. of motion for the ζ^i are:

$$\ddot{\zeta}^i = -\zeta^i \left(k_i + \sum_j \nu_{ij} |\zeta^j|^2 \right), \quad i = 1, \dots, k\tag{5.14}$$

We observe that if the range of $i = 1, \dots, k$ is partitioned into say three groups q_1, q_2, q_3 of non-negative integers, with $q_1 + q_2 + q_3 = k$ and moreover q_1 of \vec{n}_i s are equal, say \vec{n}_1 , q_2 are equal, say \vec{n}_2 and the same for q_3, \vec{n}_3 the matrix $k \times k$ ν_{ij} has a special structure and there exist only three matrix elements which we call $\nu_{12} = (\vec{n}_1 \times \vec{n}_2)^2, \nu_{23} = (\vec{n}_2 \times \vec{n}_3)^2$ as well as $\nu_{31} = (\vec{n}_3 \times \vec{n}_1)^2$. Furthermore we call $\vec{w}_1 = (\zeta^1, \zeta^2, \dots, \zeta^{q_1}), \vec{w}_2 = (\zeta^{q_1+1}, \dots, \zeta^{q_1+q_2}), \vec{w}_3 = (\zeta^{q_1+q_2+1}, \dots, \zeta^k)$ the three q_1, q_2, q_3 dimensional complex vectors. Then the eqs. of motion become

$$\begin{aligned} \ddot{w}_1 &= -\vec{w}_1(k_1 + \nu_{12}|w_2|^2 + \nu_{13}|w_3|^2) \\ k_i &= \sum_{\alpha=1}^m R_\alpha^2 (\vec{m}_\alpha \times \vec{n}_i)^2, \quad i = 1, 2, 3 \end{aligned} \quad (5.15)$$

Similarly for w_2, w_3 . With

$$|\vec{w}_1|^2 = \sum_{i=1}^{q_1} |\zeta^i|^2, \quad |\vec{w}_2|^2 = \sum_{i=q_1+1}^{q_1+q_2} |\zeta^i|^2, \quad |\vec{w}_3|^2 = \sum_{i=q_1+q_2+1}^{q_1+q_2+q_3} |\zeta^i|^2 \quad (5.16)$$

The Hamiltonian (5.12) now becomes

$$H = 2\pi T_2 \left[\sum_{i=1}^3 |\dot{w}_i|^2 + k_i |\vec{w}_i|^2 + \nu_{12} |\vec{w}_1|^2 |\vec{w}_2|^2 + \nu_{23} |\vec{w}_2|^2 |\vec{w}_3|^2 + \nu_{13} |\vec{w}_1|^2 |\vec{w}_3|^2 \right] \quad (5.17)$$

We observe that the initial $SO(2k)$ space-rotational invariance of the system is broken down to $U(q_1) \times U(q_2) \times U(q_3)$ symmetry. Note also that because of the cross product term ν_{ij} there is a modular invariance $SL(2, Z)$ which preserves ν_{ij} . The new terms $k_i |w_i|^2$ are harmonic terms which are induced by the interactions of the windings \vec{m}_α with the $e^{i\vec{n}_i \cdot \vec{\sigma}}$ dependence of the ansatz.

The conserved "complex" angular momenta for every factor of $U(q_1) \times U(q_2) \times U(q_3)$, call it generically $U(n)$, are determined from the Hamiltonian (5.12) and Noether's theorem. The generators of $U(n)$ are $n \times n$ hermitian matrices of three types. Firstly $\frac{n(n-1)}{2} T^{2(ij)}$ Hermitian matrices with elements $-i$ and i in entries (ij) and (ji) respectively with zero everywhere else. Secondly there exist $\frac{n(n-1)}{2} T^{1(ij)}$ Hermitian matrices with 1 in both (ij) and (ji) positions with zero everywhere else and lastly $n T^{3(ii)}$ with element 1 in positions (ii) and zero otherwise. For these three generators we find the conserved angular momenta

$$\begin{aligned} T_1^{(ij)} &= \frac{1}{2} \left(z^i \bar{z}^j + \bar{z}^i z^j \right) - \frac{1}{2} \left(z^j \bar{z}^i + \bar{z}^j z^i \right), \quad i > j = 1, \dots, n \\ T_2^{(ij)} &= \frac{-i}{2} \left(z^i \bar{z}^j - \bar{z}^i z^j \right) - \frac{i}{2} \left(z^j \bar{z}^i - \bar{z}^j z^i \right), \quad i > j = 1, \dots, n \\ T_3^{(ii)} &= \frac{-i}{2} \left(z^i \bar{z}^i - \bar{z}^i z^i \right), \quad i = 1, \dots, n \end{aligned} \quad (5.18)$$

These are real conserved quantities which can be grouped into one complex and one real as follows:

$$\begin{aligned} T^{(ij)} &= T_1^{(ij)} + T_2^{(ij)} = z^i \bar{z}^j - \bar{z}^j z^i, \quad i > j = 1, \dots, n \\ T_3^{(ii)} &= -\frac{i}{2} T^{(ii)} \end{aligned} \quad (5.19)$$

By using some familiar identities we demonstrate that the Casimir element

$$\sum_{i>j} \left[\left(T_1^{(ij)} \right)^2 + \left(T_2^{(ij)} \right)^2 \right] + \sum_i \left(T_3^{(ii)} \right)^2 \equiv \vec{T}^2 \quad (5.20)$$

is related to the generic kinetic term

$$|\dot{\vec{z}}|^2 = |\dot{z}_1|^2 + \dots + |\dot{z}_n|^2 = \frac{\vec{T}^2}{r^2} + \dot{r}^2 \quad (5.21)$$

where $r^2 = |z_1|^2 + \dots + |z_n|^2$. So if we call the lengths of the complex vectors $|\vec{w}^i| = r_i, i = 1, 2, 3$ and the Casimirs of each factor $U(q_i)$ $T_i^2 = \vec{T}_i^2, i = 1, 2, 3$ the Hamiltonian can be written as

$$H = 2\pi^2 T_2 \left[\sum_{i=1}^3 \left(\dot{r}_i^2 + \frac{T_i^2}{r_i^2} \right) + \sum_{i=1}^3 k_i r_i^2 + \nu_3 r_1^2 r_2^2 + \nu_2 r_1^2 r_3^2 + \nu_1 r_2^2 r_3^2 \right] \quad (5.22)$$

with $\nu_1 = \nu_{23}, \nu_2 = \nu_3, \nu_3 = \nu_{12}$.

In order to obtain the Euler-Top solutions we proceed as with the spherical cases (S^2, S^3) of ch.4. The energy minimization conditions for constant radii $r_i, i = 1, 2, 3$ are as follows:

$$T_1^2 = r_1^4 (k_1 + \nu_3 r_2^2 + \nu_2 r_3^2) \quad (5.23)$$

Similarly for the other components.

We observe that the permutation symmetry $r_1 \leftrightarrow r_2 \leftrightarrow r_3$ is broken unless we have the special point $k_1 = k_2 = k_3 = k, \nu_1 = \nu_2 = \nu_3 = \nu$. If we choose $\vec{n}_1 + \vec{n}_2 + \vec{n}_3 = 0$ (special embeddings) then we guarantee that $\nu_1 = \nu_2 = \nu_3 = \nu$. We proceed to solve (5.23) for the special point

$$T_1^2 = r_1^4 (k + \nu(r_2^2 + r_3^2)) \quad (5.24)$$

Similarly for the other components. We observe that the difference with the previous S^2 case lies in the harmonic term k . For the completely symmetric case (symmetric toroidal 2-brane Top) $T_1^2 = T_2^2 = T_3^2 = T_s^2$ and $r_1^2 = r_2^2 = r_3^2 = r_s^2$ we find

$$T_s^2 = r_s^4 (k + 2\nu r_s^2) \quad (5.25)$$

while for the axially symmetric case $r_1 = r_2 = r, T_1 = T_2 = T$ we obtain

$$\begin{aligned} T^2 &= r^4 (k + \nu(r^2 + r_3^2)) \\ T_3^2 &= r_3^4 (k + 2\nu r_3 r^2) \end{aligned} \quad (5.26)$$

For the symmetric case it is possible to get an analytic expression for the solution which follows from the careful analysis of the cubic equation (5.25). We define two ratios

$$\rho_k = \left(\frac{k}{6\nu} \right)^3, \quad \rho_T = \frac{T^2}{4\nu}. \quad (5.27)$$

For $\rho_T > 2\rho_k$ the equilibrium value of the radius of the torus which corresponds to the balancing out of the attractive tension against the repulsive angular kinetic energy is found to be

$$r_s^2 = \left[\rho_T - \rho_k + \sqrt{\rho_T(\rho_T - 2\rho_k)} \right]^{1/3} + \left[\rho_T - \rho_k - \sqrt{\rho_T(\rho_T - 2\rho_k)} \right]^{1/3} - \rho_k^{1/3} \quad (5.28)$$

while for $\rho_T < 2\rho_k$ combinations with the third root of unity $e^{2\pi i/3}$ give the result. The eq. (5.25) has always one largest positive root. The energy of the solution is

$$E_s = 8\pi^2 T_2 \nu \left(r_s^4 + 4\rho_k^{1/3} r_s^2 \right). \quad (5.29)$$

For large angular momenta $\rho_T \gg \rho_k, \rho_T \rightarrow \infty$ the radius r_s^2 behaves like

$$r_s^2 \sim \left(\frac{T^2}{4\nu} \right)^{1/3} \quad (5.30)$$

while the energy scales like

$$E_s \sim (\nu T)^{4/3} \quad (5.31)$$

We have an identical power law behaviour with the S^2 case (apart from the factor $\nu = (\vec{n}_1 \times \vec{n}_2)^2$ see rel.(5.13). The axially symmetric case (5.26) is algebraically not tractable apart from some special points in the space of parameters ν, k, T^2, T^3 . We now proceed to discuss the time dependence of the complex vectors $\vec{w}_i, i = 1, 2, 3$.

$$\vec{w}_i = e^{i\Omega_i t} \vec{w}_i(t=0). \quad (5.32)$$

In general Ω_i is a linear combination of the T_1, T_2, T_3 hermitian matrices discussed previously. By using U_{q_i} transformations we can bring $\vec{w}_i(t=0), \vec{w}_i(t=0)$ in the (z_1, z_2) complex plane and the Ω_i have the form of an $SU(2)$ hermitian matrix. In the simplest case of a diagonal $U(1)$ matrices we get angular velocities ω_i which satisfy the (5.15) eqs. of motion

$$\omega_1^2 = k_1 + v_3 r_2^2 + v_2 r_3^2 \quad (5.33)$$

Similarly for ω_2, ω_3 . We can check from (5.21)

$$T_i^2 = \omega_i^2 \cdot r_i^4, \quad i = 1, 2, 3 \quad (5.34)$$

and so the minimization conditions are identical with the eqs. of motion (5.15).

5.2 The three dimensional spinning torus T^3

Our last but not least example of spinning p-brane is the spinning T^3 torus ($p = 3$). The example is the richest one which exhibits unitary group symmetries $\prod_{i=1}^4 U(q_i)$ as well as a larger modular group symmetry $SL(3, Z)$. Moreover, it leads to the $p = 2$ (membrane) case by double dimensional reduction. This is an extension of the reduction of the membrane ($p = 2$) to the string case ($p = 1$).

We start again from the basic Hamiltonian

$$H = \frac{T_p}{2} \int d^3\xi \left[\dot{X}^{i^2} + \det(\partial_\alpha X^i \partial_\beta X^i) \right] \quad (5.35)$$

and the constraints

$$\left\{ \dot{X}^i, X^i \right\}_{\alpha, \beta} = 0 \quad \alpha \neq \beta = 1, 2, 3, \quad i = 1, \dots, D-2 \quad (5.36)$$

The volume preserving diffeomorphisms contain also global translations $P_\alpha, \alpha = 1, 2, 3$ along cycles at T^3 which are not connected to the identity. The connected subgroup is generated polynomially by the Nambu-Bracket algebra.

$$\{e_{\vec{n}_1}, e_{\vec{n}_2}, e_{\vec{n}_3}\} = i^3 \det[\vec{n}_1, \vec{n}_2, \vec{n}_3] \cdot e_{\vec{n}_1 + \vec{n}_2 + \vec{n}_3} \quad (5.37)$$

where the basic functions $e_{\vec{n}}$ are:

$$e_{\vec{n}} = e^{i\vec{n} \cdot \vec{\xi}}, \quad \vec{\xi} \in [0, 2\pi]^3, \quad \vec{n} = (n^1, n^2, n^3) \in Z^3 \quad (5.38)$$

and

$$\det[\vec{n}_1, \vec{n}_2, \vec{n}_3] = \epsilon_{\alpha\beta\gamma} n_1^\alpha n_2^\beta n_3^\gamma \quad (5.39)$$

The automorphism group of (5.37) contains the $SL(3, Z)$ modular group which leaves the structure constants invariant $\vec{n} \rightarrow A\vec{n}$, $A \in SL(3, Z)$, $A = (A_{ij})$ integer matrix with $\det A = 1$

In order to proceed with our ansatz we separate the $D-2$ target coordinates $X^i, i = 1, \dots, D-2$ into two groups. Firstly we pair X^1, X^2, \dots, X^{2k} into complex ones $2k < D-2$

$$Z^l = X^{2l-1} + iX^{2l}, \quad l = 1, \dots, k \quad (5.40)$$

and the rest $D-2-2k \equiv m, Y^\alpha, \alpha = 1, \dots, m$. The determinant $\det g_{\alpha\beta}$ of the induced metric:

$$g_{\alpha\beta} = \frac{1}{2} \partial_\alpha Z^l \partial_\beta \bar{Z}^l + \frac{1}{2} \partial_\alpha \bar{Z}^l \partial_\beta Z^l + \partial_\alpha Y^a \partial_\beta Y^a, \quad l = 1, \dots, k, \quad a = 1, \dots, m \quad \alpha, \beta = 1, 2, 3 \quad (5.41)$$

can be calculated. We derive the Hamiltonian in terms of Z^l, Y^a s:

$$\begin{aligned} H = \frac{T_3}{2} \int d^3\xi \left(|\dot{Z}^i|^2 + |\dot{Y}^a|^2 + \frac{1}{24} |\{Z^i, Z^j, Z^k\}|^2 + \right. \\ \left. \frac{1}{8} |\{Z^i, Z^j, \bar{Z}^k\}|^2 + \frac{1}{4} |\{Z^i, Z^j, Y^a\}|^2 + \frac{1}{4} |\{Z^i, \bar{Z}^j, Y^a\}|^2 + \right. \\ \left. \frac{1}{2} |\{Z^i, Y^a, Y^b\}|^2 + \frac{1}{3!} |\{Y^a, Y^b, Y^c\}|^2 \right) \end{aligned} \quad (5.42)$$

with $a, b, c = 1, \dots, m$ and the constraints:

$$\left\{ \dot{Z}^i, \bar{Z}^i \right\} + c \cdot c + \left\{ \dot{Y}^a, Y^a \right\}_{a,b} = 0, \quad a, b = 1, 2, 3 \quad (5.43)$$

We notice here that upon double dimensional reduction that is by compactification on a circle and by assuming that $Z^i, i = 1, \dots, k$ depend only on ξ^1, ξ^2 and not on ξ^3 , we can get from the Hamiltonian of $p = 3$ (5.42) toroidal of branes the $p = 2$ ones. Indeed the above assumption leads to (constant terms are neglected):

$$\begin{aligned} H = \frac{T_3}{2} 2\pi \int d^2\xi \left[|\dot{Z}^i|^2 + \frac{1}{4} \nu (|\{Z^i, Z^j\}|^2 + |\{Z^i, \bar{Z}^j\}|^2) + \frac{1}{2} |L_{a,b} Z^i|^2 \right], \\ i, j = 1, \dots, k \quad a, b = 1, \dots, m \end{aligned} \quad (5.44)$$

with

$$\begin{aligned}
 L_{a,b} &= R^a R^b \left[(m_2^a m_3^b - m_3^a m_2^b) \partial_1 + (m_3^a m_1^b - m_1^a m_3^b) \partial_2 \right] \\
 \nu &= \sum_a (R^a m_3^a)^2, \quad a, b = 1, \dots, m.
 \end{aligned}
 \tag{5.45}$$

With appropriate diagonalization and rescaling of the operator $L_{a,b}$ we can arrive at normal form of the harmonic term $|L_{a,b}\tau^i|^2$ and derive eqs. of motion for T^2 . The compactified target coordinates induce, constant, harmonic and unharmonic terms respectively on the Hamiltonian. The constant term corresponds to the KK kinetic energy as well as the winding energy $\sum_{a,b,c} (R^a R^b R^c)^2 \det^2(\vec{m}^a \vec{m}^b \vec{m}^c)$. The reduced Hamiltonian without the constant term is as follows (summation over the indices is implied):

$$\begin{aligned}
 H &= \frac{T_3}{2} (2\pi)^3 \left[|\dot{\zeta}^i|^2 + \frac{1}{6} \det^2(\vec{n}_i \vec{n}_j \vec{n}_k) |\zeta^i|^2 |\zeta^j|^2 |\zeta^k|^2 \right. \\
 &\quad \left. + \frac{1}{2} |\zeta^i|^2 |\zeta^j|^2 R^{a^2} \det^2(\vec{n}_i, \vec{n}_j, \vec{m}^a) + \frac{1}{2} |\zeta^i|^2 R^{a^2} R^{b^2} \det^2(\vec{n}_i, \vec{m}^a, \vec{m}^b) \right]
 \end{aligned}
 \tag{5.46}$$

and the ζ^i complex scale factors satisfy the eqs.:

$$\ddot{\zeta}^i = -\frac{1}{2} \zeta^i \left[\sum_{i,l \neq i} |\zeta^j|^2 |\zeta^k|^2 \lambda_{ijk} + 2 \sum_{j \neq i} |\zeta^j|^2 \nu_{ij} + k_i \right]
 \tag{5.47}$$

with

$$\begin{aligned}
 \lambda_{ijl} &= \det^2(\vec{n}_i, \vec{n}_j, \vec{n}_l), \quad i \neq j \neq l = 1, \dots, k \\
 \nu_{ij} &= \sum_{\alpha} R^{\alpha^2} \det^2(\vec{n}_i, \vec{n}_j, \vec{m}^{\alpha}) \\
 k_i &= \sum_{\alpha \neq \beta} R^{\alpha^2} R^{\beta^2} \det^2(\vec{n}_i, \vec{m}^{\alpha}, \vec{m}^{\beta})
 \end{aligned}
 \tag{5.48}$$

We now have the options to either use the ansatz of many $U(1)$ s (ref. [12])

$$\zeta^i = R^i e^{i\omega_i t}, \quad i = 1, \dots, k
 \tag{5.49}$$

or to form 4-complex vectors of $q_j, j = 1, 2, 3, 4$ components, of $q_j \zeta^i$ s $j = 1, 2, 3, 4$ which possess only four different \vec{n}_i say $q_1 \vec{n}_1$'s, $q_2 \vec{n}_2$'s and we make the ansatz:

$$\begin{aligned}
 \vec{w}_1 &= (\zeta^1, \zeta^2, \dots, \zeta^{q_1}) e^{i\vec{n}_1 \cdot \vec{\xi}} \\
 \vec{w}_2 &= (\zeta^{q_1+1}, \dots, \zeta^{q_1+q_2}) e^{i\vec{n}_2 \cdot \vec{\xi}} \\
 \vec{w}_3 &= (\zeta^{q_1+q_2+1}, \dots, \zeta^{q_1+q_2+q_3}) e^{i\vec{n}_3 \cdot \vec{\xi}} \\
 \vec{w}_4 &= (\zeta^{q_1+q_2+q_3+1}, \dots, \zeta^k) e^{i\vec{n}_4 \cdot \vec{\xi}} \\
 k &= q_1 + q_2 + q_3 + q_4
 \end{aligned}
 \tag{5.50}$$

The resulting Hamiltonian is:

$$\begin{aligned}
 H = & \frac{T_3}{2}(2\pi)^3 \left[\sum_{i=1}^4 \left(\dot{r}_i^2 + \frac{T_i^2}{r_i^2} \right) + \frac{1}{6} \sum_{i \neq j \neq k=1}^4 \lambda_{ijk} r_i^2 r_j^2 r_k^2 \right. \\
 & \left. + \frac{1}{2} \sum_{i \neq j=1}^4 \nu_{ij} r_i^2 r_j^2 + \frac{1}{2} \sum_{i=1}^4 k_i r_i^2 \right] \quad (5.51)
 \end{aligned}$$

where $r_i^2 = |\vec{w}_i|^2$, and

$$|\dot{\vec{w}}_i|^2 = \dot{r}_i^2 + \frac{T_i^2}{r_i^2}, \quad i = 1, 2, 3, 4 \quad (5.52)$$

with T_i^2 being the $U(q_i)$, $i = 1, 2, 3, 4$ Casimirs . The time dependence of the ansatz is given by:

$$\vec{w}_i(t) = e^{i\Omega_i t} \vec{w}_i(o), \quad i = 1, 2, 3, 4 \quad (5.53)$$

with Ω_i being the generators of $U(q_i)$. By diagonalizing the Ω_i as in the case of T^2 in the appropriate complex planes of \vec{w}_i s we get from the eqs of motion

$$\begin{aligned}
 \omega_1^2 = & \frac{1}{3}(\lambda_{123} r_2^2 r_3^2 + \lambda_{134} r_3^2 r_4^2) \\
 & + \nu_{12} r_2^2 + \nu_{13} r_3^2 + \nu_{14} r_4^2 + k_1 \quad (5.54)
 \end{aligned}$$

and cyclically for the other indices $i = 1, 2, 3, 4$.

These relations correspond to nothing else but the minimization conditions for the effective potential (rel.5.2.16) (since $T_i^2 = \omega_i^2 r_i^4$)

$$V_{\text{eff}} = H - \frac{(2\pi)^3}{2} T_3 \sum_{i=1}^4 r_i^2 \quad (5.55)$$

For the completely symmetric T^3 with symmetric initial conditions, (which can be satisfied if $\vec{n}_1 + \vec{n}_2 + \vec{n}_3 + \vec{n}_4 = 0$), $T_i = T, k_i = k, \nu_{ij} = \nu, \lambda_{ijk} = \lambda, i \neq j \neq k = 1, 2, 3, 4$ we obtain

$$T^2 = r^4 (\lambda r^4 + \nu r^2 + k) \quad (5.56)$$

setting $r^2 = u$ we obtain the 4rth order polynomial equation

$$T^2 = \lambda u^4 + \nu u^3 + k u^2 \quad (5.57)$$

which can be solved by quadratures . Indeed there exist two real roots for u (one positive and one negative) as well as a pair of complex conjugate roots, for any positive T^2, λ, ν, k . The energy of the configurations is expressed in terms only of λ, ν, k through the positive root u_s of (5.57) ($u_s = r_s^2$)

$$E_s = \frac{T_3}{2}(2\pi)^3 \left[\frac{14}{3} u_s^3 + 15 \nu u_s^2 + 6 k u_s \right] \quad (5.58)$$

For large values of the angular momenta $T^2 \rightarrow \infty$ the energy scales as:

$$E \sim (\lambda T^2)^{3/4} \quad (5.59)$$

6. Interpretation of the results - conclusions

We have been working in this paper with the Light Cone Gauge fixed Hamiltonian of the Nambu-Goto p-branes. The target space dimensions for various p, $p = 1, 2, 3, \dots$ are restricted by target space and k-world volume supersymmetry in order that physical bosonic and fermionic degrees to match. The relevant brane scan determines these dimensions. For $p = 1, D = 3, 4, 6, 10$ for $p = 2, D = 4, 5, 7, 11$, for $p = 3, D = 6, 8$ and $p = 4, D = 9$ and finally $p = 5, D = 10$. If one adds gauge and tensor fields on the world volume of the branes there are additional restrictions [16]. In this case the p-branes are charged under the gauge groups. For compact p-branes the total charge must be zero(Gauss-Law).

With the advent of D_p -branes [9] it was understood that there are solitonic objects of type IIA-B superstring theories in $D = 10$, where for IIA theories $p = 0, 2, 4, 6, 8$ carrying NS-NS charges and for IIB theories $p = 1, 3, 5, 7, 9$ carrying RR charges respectively. The most intriguing ones are of the IIA type $p = 2$ super D-membrane and of the type IIB $p = 3$ selfdual one along with the $p = 5$ famous fivebrane. The D-branes apart from being the sought after sources of RR and NS charges they have more degrees of freedom, the various p-form gauge world volume fields. Although so rich in structure and so well studied they have infinite extent (infinite charge and energy). Their finite (charge and energy) cousins (the Nambu-Goto p-branes) still escape our ability to describe them dynamically (unless compactified on various compact submanifolds) due to strong string coupling.

Our solutions are not charged but if we turn on the 11-dim flux field then the total charge becomes zero but with the dipole and multipole moments non-zero. In the latter case the equations of motion get modified. The simplest case is the 11dim. pp-wave background with a constant flux [5]

In a relatively recent work J.J. Rousso et.al. [15] studied rotating toroidal p-branes (Nambu-Goto ones) and observed that they represent type IIA-B solitons. Essentially the argument for type IIA is that the tension T_2 of the $p = 2$ membrane compactified from $D = 11$ dim. by double dimensional reduction to $D = 10$ on a circle of radius R_{10} goes like $T_2 = \frac{T}{g_{IIA}}$ for fixed string tension T. Similarly one has $T_2 = \frac{T}{g_{IIB}}$ for type IIB string theory which is compactified directly from $D = 11$ to $D = 9$ on a Torus $T^2 = S^1 \times S^1$ followed by a T-duality on the second S^1 . In the above work the solutions are found in a covariant gauge $X^o = P \cdot \tau$ and there are constraints which cannot be solved except for in some special cases.

In our examples (ch. 4-5) the rotating $p = 2$ solutions are given in the light-cone gauge where the constraints are solved automatically by the ansatz. The nice arguments of J.J.Rousso et.al. for the solitonic character of the $p = 2$ Toroidal membranes go through also in our case. Moreover we presented new results for S^3, T^3 spinning $p = 3$ branes. We would like to call these solutions massive giant gravitons of flat spacetimes. Our solutions are embedable in lightcone pp-wave backgrounds with fluxes. On these problems we are currently at work. More general backgrounds like $G_2, AdS^7 \times S^4$ etc. are expected to host such solutions although in these cases the constraints in general cannot be solved [14, 23, 24].

In conclusion, we have constructed new spinning $p = 2, S^2, T^2$ and $p = 3, S^3, T^3$

Nambu-Goto p-branes which behave like Euler Tops with higher rotational symmetries $\prod_i(SO(q_i))$ or $\prod_i(SU(q_i))$ respectively. This is due to the balancing out of the attractive brane-tension forces in higher dimensions by the repulsive effect of rotation alone for S^2, S^3 and in conjunction with the induced harmonic forces arising from Toroidal Compactifications for T^2, T^3 . The minimization of the energy led to its unique scaling with the angular momenta (and for T^2, T^3 from the winding). For the case of T^2 (and presumably T^3) the energy has solitonic dependence on type IIA, IIB string couplings. These solutions can be thought of as particle like objects with quantum numbers $\prod_i SO(q_i)$ for S^2, S^3 and $\prod_i U(q_i)$ for T^2, T^3 . For the case of completely symmetric configurations S^2, S^3 or T^2, T^3 (same radii in all dimensions) the equilibrium eqs. can be solved analytically by elliptic integrals as well as the angular velocities time dependence. These configurations, however, are only excitations of the Euler Tops.

One of the many interesting open questions is the Matrix model construction and the corresponding Euler-Top solutions in flat or pp-wave backgrounds for p-branes. There are several attempts [13, 14], but we think that more drastic propositions like the works of H.Awata et.al in [17] should be given more attention [25] Among possible interesting applications would be a spinning brane-world scenario for both S^3 and T^3 solutions.

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